B-Math-II Final Exam ; Analysis 3; Open Notes

Time : 2.30 hrs; Max Mark:70 ; 29 December 2021

1. a. Let $F(x,y) := x^3 + y^3 - 3xy - 4, (x,y) \in \mathbb{R}^2$. Show that there exists a continuously differentiable function g defined by the equation $F(x,y) = 0$ in a neighborhood of $x = 2$ such that $g(2) = 2$, and also find its derivative.
b. Show that the function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $F(x,y) := (y \sin x, x + y + 1)$ is locally invertible at $(0,1)$. (9+6)

2. Let $-\infty < a < b < \infty$ and let $\{x_n, n \geq 1\}$ be a sequence in (a,b). Let $c_n \in \mathbb{R}$ such that $\sum\limits_{n=1}^{\infty} |c_n| < \infty$. Let $I_A(x)$ be the indicator function of the set A. Let

$$f(x) := \sum_{n=1}^{\infty} c_n I_{(0,\infty)}(x - x_n), \ a \leq x \leq b.$$

a. Show that the series defining $f(x)$ converges uniformly in $[a,b]$.
b. Show that f is continuous at each $x \neq x_n$. (5+10)

3. Let $F : \mathbb{R} \to \mathbb{R}$ with $F(a) = b$.
a. State suitable hypothesis on F such that it has a continuously differentiable local inverse in a neighborhood of the point b. Prove your claim.
b. Define $f(t) := t + t\sin(\frac{1}{t}), t \neq 0; f(0) := 0$. Show that $f'(0) = 1$. Does this contradict your claim in part a) ? If yes, explain why. If no, explain why not.
(10 + 5)

4. Let $F(x,y) := (e^x \cos y - 1, e^x \sin y); G_1(x,y) := (e^x \cos y - 1, y); G_2(x,y) := (x, (1+x)\tan y); (x,y) \in \mathbb{R}^2$.
a. Show that $F = G_2 \circ G_1$ in some neighborhood of $(0,0)$.
b. Compute the Jacobian $DF(0,0)$ and verify the same using the chain rule.
c. Let $H_2(x,y) := (x, e^x \sin y), (x,y) \in \mathbb{R}^2$. Find $H_1(x,y) = (h(x,y), y)$ so that $F = H_1 \circ H_2$ in a neighborhood of $(0,0)$. (5+5+5)

5. Consider the 1-form $\eta := \frac{x\ dy - y\ dx}{x^2+y^2}$ in $\mathbb{R}^2 - \{(0,0)\}$. Let $\gamma(t) := (a\cos t, a\sin t), 0 \leq t \leq 2\pi, a > 0$. Let $\{\Gamma(t), t \in [0,2\pi]\}$ be a simple closed smooth curve in $\mathbb{R}^2 - \{(0,0)\}$. Let $r : T := [0,2\pi] \times [0,1] \to S := r(T)$ be the 2-surface defined by

$$r(t,u) := (1-u)\Gamma(t) + u\gamma(t).$$

Assume that r is 1-1 and $S \subset \mathbb{R}^2 - \{(0,0)\}$.
a. Show that $d\eta = 0$.
b. Describe the boundary $\partial S = r(\partial T)$.
c. Show that $\int\limits_{\Gamma} \eta = 2\pi$. (5+4+6)